

Study of the Initial Value Problems Appearing in a Factorization Method of Second Order Elliptic Boundary Value Problems

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Abstract

In [8] we presented a method to factorize a second order boundary value problem into a system of uncoupled first order initial value problems, together with a nonlinear Riccati type equation for functional operators. A weak sense was given to that system but we did not perform a direct study of those equations. This factorization utilizes either the Neumann to Dirichlet (NtD) operator or the Dirichlet to Neumann (DtN) operator, which satisfy a Riccati equation. Here we consider the framework of Hilbert-Schmidt operators, which provides tools for a direct study of this Riccati type equation. Once we have solved the system of Cauchy problems, we show that its solution solves the original second order boundary value problem. Finally, we indicate how this techniques can be used to find suitable transparent conditions.

keywords: Factorization, boundary value problem, Hilbert-Schmidt operator, Riccati equation; invariant embedding; Neumann-to-Dirichlet (NtD) operator; Dirichlet-to-Neumann (DtN) operator, transparent conditions.

1 Introduction

In [1] Angel and Bellman proposed a method based on spatial invariant embedding to transform a second order elliptic boundary value problem in a rectangle in a system of first

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order decoupled initial value problems which can be solved by a two sweep process. In [8] Henry and Ramos gave a complete justification for this transform for the Poisson equation in a n -dimensional cylindrical domain. The invariant embedding was performed using the coordinate along the axis of the cylinder. The Neumann-to-Dirichlet (NtD) operator on a section of the cylinder was shown to satisfy a Riccati equation. The relationship of this method with the similar one to derive the optimal feedback for optimal control problems of parabolic equations was described. The study of the well-posedness of the Riccati equation was similar to the one used by Lions [9] and based on a Galerkin method. Furthermore the relationship between this factorization method and the block LU factorization for a discretized version of the problem was established.

In this paper we consider a similar problem, allowing for a diffusivity coefficient depending on the coordinate along the axis of the cylinder. Following the paper by Temam [13] we study directly the Riccati equation, that is without reference to the boundary value problem (as it was done in [8]), in a Hilbert-Schmidt framework. We cannot apply the results of [13] as here the operator appearing in the quadratic term is unbounded (the same difficulty was encountered in [8]). The Hilbert-Schmidt framework appears to be efficient for the studied problem, using a fixed basis of eigenfunctions of the Laplacian restricted to the section. Although it defines the solution in a weaker sense than the one that could be hoped, this regularity is easily recovered. The solution of the boundary value problem is then obtained in factorized form, that is by solving two uncoupled Cauchy problems. This technique can be used to find suitable transparent boundary condition in a subdomain.

In section 2 we recall the formal derivation of the factorization. We recall some Hilbert-Schmidt properties from [13] in section 3. The study of the Riccati equation is done in section 4. The factorized form of the boundary value problem is obtained in section 5. The initial definition of P as a NtD operator given in [8] is recovered and its use to build transparent boundary conditions is presented.

2 The Factorization Method for Boundary Value Problems

Let Ω be the cylinder $\Omega =]0, a[\times \mathcal{O}$ in \mathbb{R}^d , with \mathcal{O} a bounded open set in \mathbb{R}^{d-1} and $\Gamma_s = \{s\} \times \mathcal{O}$. The lateral boundary of the cylinder is denoted by $\Sigma =]0, a[\times \partial\mathcal{O}$ and a general point $(x_1, x_2, \dots, x_d) \in \Omega$ is also denoted by (x, y) , where $x = x_1$ and y denotes the independent variables (x_2, \dots, x_d) . We use the Hilbert space $H_{00}^{1/2}(\mathcal{O}) = [H_0^1(\mathcal{O}), L^2(\mathcal{O})]_{1/2}$ defined in [10]. Let $f \in L^2(\Omega)$, $y_0 \in H_{00}^{1/2}(\mathcal{O})$, $y_a \in H_{00}^{1/2}(\mathcal{O})'$ (respectively $y_a \in H_{00}^{1/2}(\mathcal{O})$) and $\alpha \in L^\infty(0, a)$ (the study of the particular case with $\alpha \in \mathbb{R}$ constant can be seen in [4]) with

$$0 < \bar{\alpha} = \inf_{x \in [0, a]} \alpha(x) \leq \underline{\alpha} = \sup_{x \in [0, a]} \alpha(x).$$

Let us consider the problem

$$(1) \quad \begin{cases} -\nabla \cdot (\alpha(x) \nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Sigma, \\ u = u_0 & \text{on } \Gamma_0, \\ \alpha \frac{\partial u}{\partial x} = u_a & \text{on } \Gamma_a \end{cases} \quad (\text{respectively } u = u_a \text{ on } \Gamma_a).$$

Let us embed this problem in a family of similar problems defined in sub-cylinders $\Omega_s =]0, s[\times \mathcal{O}$ for $0 < s < a$, with the boundary data $\alpha \frac{\partial u}{\partial x} = h$ on Γ_s . This family depends on s and h . By linearity, we have the existence of an operator P and a function r satisfying

$$u|_{\Gamma_s} = P(s)h + r(s).$$

For a solution of (1) this identity is verified for any x :

$$u(x) = P(x)\alpha(x)\frac{\partial u}{\partial x}(x) + r(x).$$

From now on we shall consider u as a function of x with values in a function space defined on \mathcal{O} , so we use the notation $\frac{d}{dx}$ for the x -derivation. By formal derivation (to be justified later), we have

$$\begin{aligned} \frac{du}{dx} &= \frac{dP}{dx}\alpha(x)\frac{du}{dx} + P\frac{d}{dx}\left(\alpha(x)\frac{du}{dx}\right) + \frac{dr}{dx} \\ &= \frac{dP}{dx}\alpha(x)\frac{du}{dx} - P\alpha\Delta_y P\alpha\frac{du}{dx} - P\alpha\Delta_y r - Pf + \frac{dr}{dx}. \end{aligned}$$

Therefore

$$\left(\frac{dP}{dx} - \alpha P\Delta_y P - \alpha^{-1}I\right)\alpha\frac{du}{dx} + \left(\frac{dr}{dx} - \alpha P\Delta_y r - Pf\right) = 0.$$

Then, we have

$$(2) \quad \begin{cases} \frac{dP}{dx} - \alpha P\Delta_y P - \alpha^{-1}I = 0, & P(0) = 0, \\ \frac{dr}{dx} - \alpha P\Delta_y r = Pf, & r(0) = u_0, \\ -P\alpha\frac{du}{dx} + u = r, & u(a) = P(a)u_a + r(a) \text{ (respectively } u(a) = u_a). \end{cases}$$

We point out that we have factorized the equation appearing in (1) as

$$(3) \quad \left(\frac{d}{dx} - \alpha P\Delta_y\right)\left(I - \alpha P\frac{d}{dx}\right)u = Pf,$$

which is equivalent, by using that $\frac{d}{dx}P = \frac{dP}{dx} + P\frac{d}{dx}$ and the Riccati equation to

$$\left(\frac{d}{dx} + P^{-1}\alpha^{-1}\right)\left(P^{-1} - \alpha\frac{d}{dx}\right)u = f,$$

or, in self-adjoint form,

$$\left(\frac{d}{dx}\alpha^{1/2} + P^{-1}\alpha^{-1/2}\right)\left(\alpha^{-1/2}P^{-1} - \alpha^{1/2}\frac{d}{dx}\right)u = f.$$

Remark that P^{-1} is not defined at 0.

Remark 1 Operator P is a Neumann to Dirichlet operator (NtD). We could have also used a Dirichlet to Neumann operator, which gives the system

$$\begin{cases} -\frac{dQ}{dx} + \alpha^{-1}Q^2 - \alpha\Delta_y = 0, & Q(a) = 0, \\ \frac{dw}{dx} - \alpha^{-1}Qw = f, & w(a) = u_a, \\ \alpha\frac{du}{dx} + Qu = -w, & u(0) = u_0. \end{cases}$$

We point out that we have factorized the equation appearing in (1) as

$$-\left(\frac{d}{dx} - \alpha^{-1}Q\right)\left(\alpha\frac{d}{dx} + Q\right)u = f,$$

or, in self-adjoint form,

$$-\left(\frac{d}{dx} - \alpha^{-1}Q\right)\alpha\left(\frac{d}{dx} + \alpha^{-1}Q\right)u = f.$$

In the following sections we carry out a rigorous mathematical study of system (2). In contrast with other studies of this system (see [8]) this study will be done directly, without considering the fact that those equations come from a factorization method. We show existence and uniqueness results for the solution of system (2) and we show that we recover the solution of problem (1).

3 Hilbert-Schmidt Spaces

3.1 Generalities

In this section we recall the definitions and properties of Hilbert-Schmidt spaces (see [2] and [5]) following the notation given in [13]. We consider two Hilbert spaces X, Y with scalar products $(\cdot, \cdot)_X, (\cdot, \cdot)_Y$ and associated norms $\|\cdot\|_X, \|\cdot\|_Y$. We suppose that X and Y are separable and $\{\varphi_i\}_{i=1}^\infty, \{\psi_i\}_{i=1}^\infty$ are any orthonormal basis of X and Y respectively.

Definition 2 $P \in \mathcal{L}(X, Y)$ (a bounded linear operator from X to Y) is said to be a Hilbert-Schmidt operator in case

$$\sum_{i=1}^{\infty} \|P(\varphi_i)\|_Y^2 < \infty.$$

In this case we denote $P \in X \hat{\otimes}_2 Y$ and

$$\|P\|_{X \hat{\otimes}_2 Y} = \left(\sum_{i=1}^{\infty} \|P(\varphi_i)\|_Y^2 \right)^{1/2}$$

is called the Hilbert-Schmidt norm of P .

The proof of the following properties and other results regarding Hilbert-Schmidt spaces can be seen in [5] and [2].

Lemma 3 The Hilbert-Schmidt norm is independent of the orthonormal basis used in its definition and

$$\|P\|_{X \hat{\otimes}_2 Y} = \left(\sum_{i,j=1}^{\infty} |(P\varphi_i, \psi_j)_Y|^2 \right)^{1/2}.$$

Proposition 4 Every Hilbert-Schmidt operator is compact.

Proposition 5 $X \hat{\otimes}_2 Y$ is a Hilbert space with the inner product

$$[P, Q]_{X \hat{\otimes}_2 Y} = \sum_{i=1}^{\infty} (P\varphi_i, Q\varphi_i)_Y.$$

3.2 A special case

Following [10], let us suppose that H, V are two real separable Hilbert spaces such that $V \subset H$ and V is dense in H with continuous injection. Then, we can identify H' (topological dual of H) with H and with a dense subset of V' . Therefore,

$$V \subset H \subset V',$$

with every space being dense in the following one with continuous injections.

Let us suppose that the injection $V \subset H$ is compact and let $\Lambda : V \rightarrow V'$ be the canonical isomorphism between V and V' , i.e.

$$(\Lambda u, v)_{V' \times V} = (u, v)_V \quad \forall u, v \in V.$$

Then, $\Lambda^{-1} : H \rightarrow V$ is a compact self-adjoint linear operator. Therefore (see, for instance, Theorems VI.8 and VI.11 of [3]), there exists an orthonormal basis $\{w_i\}_{i=1}^{\infty}$ of H , consisting of eigenvectors of Λ^{-1} such that

$$\Lambda w_i = \lambda_i w_i, \quad \forall i \in \mathbb{N}$$

with $\lambda_i > 0$ and

$$\lim_{i \rightarrow \infty} \lambda_i = +\infty.$$

Now,

$$\|w_i\|_V^2 = (w_i, w_i)_V = (\Lambda w_i, w_i)_{V' \times V} = (\lambda_i w_i, w_i)_H = \lambda_i \quad \forall i = 1, \dots, \infty.$$

Therefore,

$$u = \sum_{i=1}^{\infty} u_i w_i \in V \Leftrightarrow \|u\|_V^2 = \sum_{i=1}^{\infty} \lambda_i |u_i|^2 < \infty.$$

Furthermore, if $u = \sum_{i=1}^{\infty} u_i w_i \in V$ and $v = \sum_{i=1}^{\infty} v_i w_i \in V$, then

$$(u, v)_V = \sum_{i=1}^{\infty} \lambda_i u_i v_i.$$

Definition 6 For every $r \in \mathbb{R}$ we define the Hilbert space V^r in the following way:

$$u = \sum_{i=1}^{\infty} u_i w_i \in V^r \Leftrightarrow \|u\|_{V^r}^2 = \sum_{i=1}^{\infty} \lambda_i^r |u_i|^2 < \infty.$$

Furthermore, if $u = \sum_{i=1}^{\infty} u_i w_i \in V^r$ and $v = \sum_{i=1}^{\infty} v_i w_i \in V^r$, then

$$(u, v)_{V^r} = \sum_{i=1}^{\infty} \lambda_i^r u_i v_i.$$

Remark 7 We have that $H = V^0$, $V = V^1$ and $V' = V^{-1}$. Furthermore, $\{\lambda_i^{-r/2} w_i\}_{i=1}^{\infty}$ is an orthonormal basis of V^r , for all $r \in \mathbb{R}$.

Definition 8 Let $\mathcal{H} = \left\{ \sum_{\text{finite}} \mu_j w_j, \mu_j \in \mathbb{R} \right\}$ and $\langle w_j \rangle = \{\mu w_j, \mu \in \mathbb{R}\}$ for all $j \in \mathbb{N}$.

Given $i, j \in \mathbb{N}$, we consider the operator $w_i \otimes w_j : \mathcal{H} \rightarrow \langle w_j \rangle$ defined by

$$w_i \otimes w_j(\varphi) = (w_i, \varphi)_H w_j \quad \forall \varphi \in \mathcal{H}.$$

Now, since \mathcal{H} is a dense subset of V^r , for all $r \in \mathbb{R}$, we can consider the extension $w_i \otimes w_j : V^r \rightarrow \langle w_j \rangle$ defined by

$$w_i \otimes w_j(\varphi) = \varphi_i w_j \quad \forall \varphi = \sum_{k=1}^{\infty} \varphi_k w_k \in V^r.$$

Remark 9 From Definition 2 it is easy to see that, for all $i, j \in \mathbb{N}$, we have that $w_i \otimes w_j \in V^r \widehat{\otimes}_2 V^s$ for all $r, s \in \mathbb{R}$. Furthermore, if $P = \sum_{i,j=1}^{\infty} \xi_{ij} w_i \otimes w_j$, then

$$P \in V^r \widehat{\otimes}_2 V^s \Leftrightarrow \|P\|_{V^r \widehat{\otimes}_2 V^s}^2 = \sum_{i=1}^{\infty} \|P(\lambda_i^{-r/2} w_i)\|_{V^s}^2 < \infty.$$

Now,

$$P(\lambda_i^{-r/2} w_i) = \sum_{k,j=1}^{\infty} \xi_{kj} (w_k, \lambda_i^{-r/2} w_i)_H w_j = \sum_{j=1}^{\infty} \xi_{ij} \lambda_i^{-r/2} w_j.$$

Therefore

$$\|P(\lambda_i^{-r/2} w_i)\|_{V^s}^2 = \sum_{j=1}^{\infty} \xi_{ij}^2 \lambda_i^{-r} \lambda_j^s$$

and we deduce that

$$P \in V^r \widehat{\otimes}_2 V^s \Leftrightarrow \|P\|_{V^r \widehat{\otimes}_2 V^s}^2 = \sum_{i,j=1}^{\infty} \xi_{ij}^2 \lambda_i^{-r} \lambda_j^s < \infty.$$

Finally, if $Q = \sum_{i,j=1}^{\infty} \eta_{ij} w_i \otimes w_j \in V^r \widehat{\otimes}_2 V^s$, then

$$[P, Q]_{V^r \widehat{\otimes}_2 V^s} = \sum_{i=1}^{\infty} (P(\lambda_i^{-r/2} w_i), Q(\lambda_i^{-r/2} w_i))_{V^s} = \sum_{i,j=1}^{\infty} \xi_{ij} \eta_{ij} \lambda_i^{-r} \lambda_j^s.$$

Example 10 Let $P = \sum_{i,j=1}^{\infty} \xi_{ij} w_i \otimes w_j$ and $Q = \sum_{i,j=1}^{\infty} \eta_{ij} w_i \otimes w_j$.

1. $P \in H \widehat{\otimes}_2 H \Leftrightarrow \|P\|_{H \widehat{\otimes}_2 H}^2 = \sum_{i,j=1}^{\infty} \xi_{ij}^2 < \infty$ and $[P, Q]_{H \widehat{\otimes}_2 H} = \sum_{i,j=1}^{\infty} \xi_{ij} \eta_{ij}$.
2. $P \in V' \widehat{\otimes}_2 H \Leftrightarrow \|P\|_{V' \widehat{\otimes}_2 H}^2 = \sum_{i,j=1}^{\infty} \xi_{ij}^2 \lambda_i < \infty$ and $[P, Q]_{V' \widehat{\otimes}_2 H} = \sum_{i,j=1}^{\infty} \xi_{ij} \eta_{ij} \lambda_i$.
3. $P \in V \widehat{\otimes}_2 H \Leftrightarrow \|P\|_{V \widehat{\otimes}_2 H}^2 = \sum_{i,j=1}^{\infty} \xi_{ij}^2 \lambda_i^{-1} < \infty$ and $[P, Q]_{V \widehat{\otimes}_2 H} = \sum_{i,j=1}^{\infty} \xi_{ij} \eta_{ij} \lambda_i^{-1}$.
4. $P \in H \widehat{\otimes}_2 V \Leftrightarrow \|P\|_{H \widehat{\otimes}_2 V}^2 = \sum_{i,j=1}^{\infty} \xi_{ij}^2 \lambda_j < \infty$ and $[P, Q]_{H \widehat{\otimes}_2 V} = \sum_{i,j=1}^{\infty} \xi_{ij} \eta_{ij} \lambda_j$.
5. $P \in H \widehat{\otimes}_2 V' \Leftrightarrow \|P\|_{H \widehat{\otimes}_2 V'}^2 = \sum_{i,j=1}^{\infty} \xi_{ij}^2 \lambda_j^{-1} < \infty$ and $[P, Q]_{H \widehat{\otimes}_2 V'} = \sum_{i,j=1}^{\infty} \xi_{ij} \eta_{ij} \lambda_j^{-1}$.

3.3 Our particular case

In this work we shall consider the spaces $V^r \widehat{\otimes}_2 V^s$, $r, s \in \mathbb{R}$, for the particular case of $H = L^2(\mathcal{O})$ and $V = H_0^1(\mathcal{O})$, Λ being now $-\Delta$ which is an isomorphism from V to V' . Therefore, we consider as orthonormal basis of H the set $\{w_1, \dots, w_n, \dots\}$

$$\begin{cases} -\Delta_y w_n = \lambda_n w_n & \text{in } \mathcal{O} \\ w_n = 0 & \text{in } \partial\mathcal{O}, \end{cases}$$

with $\|w_i\|_{L^2(\mathcal{O})} = 1$, for all $i = 1, 2, \dots$. It has the the following properties:

$$\begin{cases} (a) (w_n, w_m)_{L^2(\mathcal{O})} = \delta_{n,m} \quad \forall m, n. \\ (b) (w_n, w_m)_{H_0^1(\mathcal{O})} = \int_{\mathcal{O}} \nabla_y w_n(y) \nabla_y w_m(y) dy = \lambda_n \delta_{n,m}, \\ (c) \left\{ \sum_{\text{finite}} \mu_j w_j, \mu_j \in \mathbb{R} \right\} \text{ is a dense subset of } V. \end{cases}$$

If we write the sequence $\{\lambda_1, \dots, \lambda_n, \dots\}$ in a nondecreasing way, it can be proved (see, for instance, Theorem VI.8 of [3]) that $\lambda_i \geq 0$, for all $i = 1, \dots, +\infty$ and

$$\lim_{i \rightarrow \infty} \lambda_i = +\infty.$$

Proposition 11 *The identity operator $I : H \rightarrow H$ can be expressed as*

$$I = \sum_{i=1}^{\infty} w_i \otimes w_i,$$

with this series converging in $\mathcal{L}(H, H)$.

Proof. Let $\varphi \in H$, then

$$\sum_{i=1}^{\infty} w_i \otimes w_i(\varphi) = \sum_{i=1}^{\infty} (\varphi, w_i)_H w_i = \sum_{i=1}^{\infty} \varphi_i w_i = \varphi,$$

where φ_i , $i = 1, \dots, \infty$ are the coordinates with respect to the orthonormal basis $\{w_i\}_{i=1}^{\infty}$ in $L^2(\mathcal{O})$. \square

Now the following question arises: Is the identity operator in some of the Hilbert-Schmidt spaces described above (as $H \widehat{\otimes}_2 H$ or $V \widehat{\otimes}_2 H$)?

It is clear that $I \notin H \widehat{\otimes}_2 H$, since (see Remark 9)

$$\sum_{i=1}^{\infty} 1 = \infty.$$

To see if the identity operator I is in some of the other Hilbert-Schmidt spaces we shall use the following lemma.

Lemma 12 (Weyl's Estimate) *Given a regular domain $\Omega \subset \mathbb{R}^n$, the asymptotic behavior of the eigenvalues for the Laplace operator with homogeneous Dirichlet boundary conditions is the following:*

$$\lambda_k \sim \frac{4\pi}{(\text{Vol}(\Omega))^{2/n}} k^{2/n}.$$

Proof. See [14]. \square

We point out that $w_i \in V^s$ for all $s \in \mathbb{R}$ and $i \in \mathbb{N}$. Further, $V^0 = H$ and $V^1 = V$. Depending on the domain dimension, we shall need to work on slightly more regular spaces than H and V .

Definition 13 *Let $\delta = 0$ if $\dim(\mathcal{O}) = 1$ and $\delta > 0$ if $\dim(\mathcal{O}) = 2$ ($\dim(\mathcal{O}) > 2$ is not of interest in applications). Let us define $\tilde{H} = V^\delta$ and $\tilde{V} = V^{1+\delta}$ (if $\dim(\mathcal{O}) = 1$, then $\tilde{H} = H$ and $\tilde{V} = V$). For $\delta \geq 0$ small enough, we have $\tilde{V} \subset V \subset \tilde{H} \subset H$.*

Corollary 14 *If $\dim(\mathcal{O}) \leq 2$, then $I \in (\tilde{V} \hat{\otimes}_2 H) \cap (H \hat{\otimes}_2 \tilde{V}')$.*

Proof. We have to prove (see Remark 9) that

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i^{1+\delta}} < +\infty,$$

which is a consequence of Lemma 12. \square

4 A direct Study of the Riccati Initial Value Problem

In [13], the author study a general equation

$$\begin{cases} \frac{dP}{dx} + PA + A^*P + \Phi(P) = F, \\ P(0) = P_0, \end{cases}$$

with $A : V \rightarrow V'$ coercive and $F \in L^2(0, a; H \hat{\otimes}_2 H)$. It is showed the existence of a solution $u \in \mathcal{C}([0, a] : H \hat{\otimes}_2 H)$. Let us study the Riccati equation

$$(4) \quad \begin{cases} \frac{dP}{dx} - \alpha P \Delta_y P = \alpha^{-1} I \quad \text{a.e. for } x \in (0, a), \\ P(0) = 0. \end{cases}$$

In problem (4), we do not have a coercive operator A and we have seen that $\alpha^{-1} I \notin H \hat{\otimes}_2 H$. We only have $\alpha^{-1} I \in (\tilde{V} \hat{\otimes}_2 H) \cap (H \hat{\otimes}_2 \tilde{V}')$ at least if $\dim(\mathcal{O}) \leq 2$. Furthermore $-\Delta_y$ is unbounded on H , so results of [13] cannot be applied directly. We shall try to get a solution $P \in \mathcal{C}([0, \infty) : V^r \hat{\otimes}_2 V^s)$, for some $r, s \in \mathbb{R}$ to be determined, by a semi-discretization and passing to the limit process.

4.1 Semi discretization

4.1.1 Formulation of the problem and monotonicity results

We approximate any space V^r , with $r \in \mathbb{R}$ by $\mathcal{V}^m = \text{span}(w_1, \dots, w_m)$ and define the m -approximate solution of (4) by the finite dimensional operator

$$P^m(x) = \sum_{i,j=1}^m \xi_{ij}^m(x) w_i \otimes w_j \in \mathcal{V}^m \widehat{\otimes}_2 \mathcal{V}^m,$$

where $\xi_{ij}^m(x)$ are chosen such that

$$(5) \quad \begin{cases} \left[\frac{dP^m}{dx} - \alpha P^m \Delta_y P^m - \alpha^{-1} I^m, w_i \otimes w_j \right]_{H \widehat{\otimes}_2 H} = 0, \quad \forall i, j \in \{1, \dots, m\} \\ P^m(0) = 0, \end{cases}$$

with $I^m = \sum_{i=1}^m w_i \otimes w_i$. We point out that $\xi_{ij}^m(x) = (P^m(x) w_i, w_j)_H$.

System (5) is a nonlinear system of equations in ξ_{ij}^m of the form:

$$\begin{cases} \frac{d\xi^m}{dx} + A^m(\xi^m) + b^m = 0, \\ \xi^m(0) = 0, \end{cases}$$

where

$$A^m(\xi^m)_{ij}(x) = -\alpha(x) \left[\left(\sum_{k,l=1}^m \xi_{kl}^m w_k \otimes w_l \right) \Delta_y \left(\sum_{r,s=1}^m \xi_{rs}^m w_r \otimes w_s \right), w_i \otimes w_j \right]_{H \widehat{\otimes}_2 H}$$

and

$$b_{ij}^m(x) = \begin{cases} -\alpha^{-1}(x), & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to verify that

$$A^m(\xi^m)_{ij} = -\alpha \left[\left(\sum_{k,l=1}^m \xi_{kl}^m w_k \otimes w_l \right) \left(\sum_{r,s=1}^m -\lambda_s \xi_{rs}^m w_r \otimes w_s \right), w_i \otimes w_j \right]_{H \widehat{\otimes}_2 H}.$$

Now, taking into account that

$$w_k \otimes w_l \circ w_r \otimes w_s = \begin{cases} w_r \otimes w_l, & \text{if } k = s, \\ 0, & \text{otherwise,} \end{cases}$$

we have that

$$A^m(\xi^m)_{ij} = \alpha \left[\sum_{k,l,r=1}^m \lambda_k \xi_{kl}^m \xi_{rk}^m w_r \otimes w_l, w_i \otimes w_j \right]_{H \widehat{\otimes}_2 H} = \alpha \sum_{k=1}^m \lambda_k \xi_{ik}^m \xi_{kj}^m.$$

Then, if we write the (i, j) -coordinates in $m \times m$ matrix form, we obtain

$$(6) \quad \begin{cases} \frac{d\xi^m}{dx} + \alpha \xi^m \Lambda^m \xi^m - \alpha^{-1} I^m = 0, \\ \xi^m(0) = 0, \end{cases}$$

where I^m represents here the $m \times m$ identity matrix and Λ^m is a diagonal matrix, with λ_i , $i = 1, \dots, m$, being the elements of the diagonal.

But this derivation is still formal since we have assumed the derivability of P^m . Then, by the theory of ordinary differential equations (Carathéodory Theorem), we know that there exists a unique local solution P^m to (5) in $[0, \delta]$, with δ small enough, which is absolutely continuous from $[0, \delta]$ to $\mathcal{V}^m \hat{\otimes}_2 \mathcal{V}^m$ (see, for instance, Theorems 5.1 and 5.3 of pages 28 and 30 of [7]). If $\alpha \in \mathcal{C}([0, a])$ we have $P^m \in \mathcal{C}^1([0, \delta] : \mathcal{V}^m \hat{\otimes}_2 \mathcal{V}^m)$.

The uniqueness of solution is also a consequence of Corollary 2 of page 13 of [12].

To go further, we need estimates on $\xi^m(s)$ (i.e. on $P^m(s)$) independent of s , in order to be able to extend the solution to $[0, \infty)$ (see, for instance, Theorem 5.2 of page 29 of [7]).

Following page 11 of [12], the solution of the above problem can be expressed as

$$\xi^m(x) = V(x)U(x)^{-1},$$

where $Y = \begin{pmatrix} U \\ V \end{pmatrix}$ satisfies the corresponding linear (Hamiltonian) matrix differential system

$$(7) \quad \begin{cases} \mathcal{J}Y'(x) + \mathcal{U}(x)Y(x) = 0 \\ Y(0) = \begin{pmatrix} I^m \\ 0 \end{pmatrix}, \end{cases}$$

with

$$\mathcal{J} = \begin{pmatrix} 0 & -I^m \\ I^m & 0 \end{pmatrix}, \quad \mathcal{U}(x) = \begin{pmatrix} \alpha^{-1}(x)I^m & 0 \\ 0 & -\alpha(x)\Lambda^m \end{pmatrix}.$$

In order to study the global existence of solution, let us suppose first that $\alpha \in \mathbb{R}$ is a constant. Therefore,

$$Y(x) = e^{Ax} \begin{pmatrix} I^m \\ 0 \end{pmatrix}, \quad \forall x \in [0, +\infty),$$

with

$$A = -\mathcal{J}^{-1}\mathcal{U} = \begin{pmatrix} 0 & -I^m \\ I^m & 0 \end{pmatrix} \begin{pmatrix} \alpha^{-1}I^m & 0 \\ 0 & -\alpha\Lambda^m \end{pmatrix} = \begin{pmatrix} 0 & \alpha\Lambda^m \\ \alpha^{-1}I^m & 0 \end{pmatrix}.$$

The eigenvalues of A are $\sqrt{\lambda_i}$ and $-\sqrt{\lambda_i}$, $i = 1, \dots, m$, and the matrix of eigenvectors is

$$\begin{pmatrix} \alpha(\Lambda^m)^{1/2} & -\alpha(\Lambda^m)^{1/2} \\ I^m & I^m \end{pmatrix}.$$

Then U and V are given by

$$(8) \quad U(x) = \frac{1}{2} \left(e^{(\Lambda^m)^{1/2}x} + e^{-(\Lambda^m)^{1/2}x} \right),$$

$$(9) \quad V(x) = \frac{(\Lambda^m)^{-1/2}}{2\alpha} \left(e^{(\Lambda^m)^{1/2}x} - e^{-(\Lambda^m)^{1/2}x} \right).$$

Thus, U is non-singular and we can extend the solution of (5) (when α is constant) from $[0, \delta]$ to $[0, +\infty]$.

Remark 15 The matrix A has positive and negative eigenvalues. The initial value problem for the Hamiltonian system (7) is well posed in finite dimension. This would not be the case for the corresponding infinite dimensional initial value problem because of the part of system (7) corresponding to positive eigenvalues $\{\sqrt{\lambda_i}\}_{i=0}^\infty$ which are going to $+\infty$. For this part of the system we have a situation similar to a backward heat equation whose ill-posedness is well-known.

Let us consider

$$\underline{P}^m(x) = \sum_{i,j=1}^m \xi_{ij}^m(x) w_i \otimes w_j \in \mathcal{V}^m \hat{\otimes}_2 \mathcal{V}^m$$

and

$$\overline{P}^m(x) = \sum_{i,j=1}^m \bar{\xi}_{ij}^m(x) w_i \otimes w_j \in \mathcal{V}^m \hat{\otimes}_2 \mathcal{V}^m,$$

the particular cases of $\alpha \equiv \underline{\alpha}$ and $\alpha \equiv \bar{\alpha}$ respectively. We have proved above that \underline{P}^m and \overline{P}^m are defined over $[0, +\infty]$.

Proposition 16 For a given $m \in \mathbb{N}$, we have that, for all $x \geq 0$,

$$\underline{P}^m(x) = \sum_{i=1}^m \xi_{ii}^m(x) w_i \otimes w_i,$$

$$\overline{P}^m(x) = \sum_{i=1}^m \bar{\xi}_{ii}^m(x) w_i \otimes w_i,$$

and there exists $\underline{P}_\infty^m, \overline{P}_\infty^m \in \mathcal{V}^m \hat{\otimes}_2 \mathcal{V}^m$ such that, if $0 \leq t_1 \leq t_2$, then

$$0 \leq \underline{P}^m(t_1) \leq \underline{P}^m(t_2) \leq \underline{P}_\infty^m,$$

$$0 \leq \overline{P}^m(t_1) \leq \overline{P}^m(t_2) \leq \overline{P}_\infty^m,$$

Proposition 17 For a given $m \in \mathbb{N}$, we can extend the solution P^m of (5) (with α not necessarily constant) from $[0, \delta]$ to $[0, a]$. Further, for all $x \in [0, a]$,

$$P^m(x) = \sum_{i=1}^m \xi_{ii}^m(x) w_i \otimes w_i \in W^{1,\infty}(0, \delta : \mathcal{V}^m \hat{\otimes}_2 \mathcal{V}^m),$$

and for all $x \in [0, a]$ we have that $0 \leq \underline{P}^m(x) \leq P^m(x) \leq \overline{P}^m(x)$.

4.1.2 Proof of Propositions 16 and 17

Proof Proposition 16

From (8)-(9) it is easy to see that \underline{P}^m and \overline{P}^m are diagonal and given by

$$\underline{\xi}_{ii}^m(x) = \frac{1}{\underline{\alpha}\sqrt{\lambda_i}} \tanh(\sqrt{\lambda_i}x), \quad i = 1, \dots, m$$

and

$$\overline{\xi}_{ii}^m(x) = \frac{1}{\overline{\alpha}\sqrt{\lambda_i}} \tanh(\sqrt{\lambda_i}x), \quad i = 1, \dots, m.$$

They can be also computed as the solution of the diagonal systems

$$\begin{cases} \frac{d\underline{\xi}_{ii}^m}{dx} + \underline{\alpha}\lambda_i(\underline{\xi}_{ii}^m)^2 - \underline{\alpha}^{-1} = 0, \\ \underline{\xi}_{ii}^m(0) = 0, \end{cases} \quad i = 1, \dots, m,$$

$$\begin{cases} \frac{d\overline{\xi}_{ii}^m}{dx} + \overline{\alpha}\lambda_i(\overline{\xi}_{ii}^m)^2 - \overline{\alpha}^{-1} = 0, \\ \overline{\xi}_{ii}^m(0) = 0, \end{cases} \quad i = 1, \dots, m.$$

Now,

$$(10) \quad \frac{d\underline{\xi}_{ii}^m}{dx}(x) = \frac{\underline{\alpha}^{-1}}{\cosh^2(\sqrt{\lambda_i}x)} > 0, \quad \forall x \in \mathbb{R},$$

$$(11) \quad \frac{d\overline{\xi}_{ii}^m}{dx}(x) = \frac{\overline{\alpha}^{-1}}{\cosh^2(\sqrt{\lambda_i}x)} > 0, \quad \forall x \in \mathbb{R},$$

and we can define

$$\underline{\xi}_{\infty,ii}^m := \lim_{x \rightarrow \infty} \underline{\xi}_{ii}^m(x) = \frac{1}{\underline{\alpha}\sqrt{\lambda_i}}, \quad i = 1, \dots, m$$

and

$$\overline{\xi}_{\infty,ii}^m := \lim_{x \rightarrow \infty} \overline{\xi}_{ii}^m(x) = \frac{1}{\overline{\alpha}\sqrt{\lambda_i}}, \quad i = 1, \dots, m,$$

which concludes the proof. \square

Proof of Proposition 17

The unique solution of (5) is the same as the solution of the diagonal system

$$(12) \quad \begin{cases} \frac{d\xi_{ii}^m}{dx} + \alpha(x)\lambda_i(\xi_{ii}^m)^2 - \alpha^{-1}(x) = 0, \\ \xi_{ii}^m(0) = 0, \end{cases} \quad i = 1, \dots, m.$$

Further, since

$$\underline{\alpha}^{-1} - \underline{\alpha}\lambda_i\xi^2 \leq \alpha^{-1}(x) - \alpha(x)\lambda_i\xi^2 \leq \overline{\alpha}^{-1} - \overline{\alpha}\lambda_i\xi^2, \quad \forall \xi \in \mathbb{R},$$

we have that (see, for instance, Theorem 6.1 of page 51 of [7])

$$\underline{\xi}_{ii}^m(x) \leq \xi_{ii}^m(x) \leq \bar{\xi}_{ii}^m(x), \quad \forall x \in [0, a],$$

which concludes the proof by applying, for instance, the continuation Theorem 5.2 of page 29 of [7]. \square

In Figure 1 we show the graphs of $\xi_{ii}^m(x)$, $\underline{\xi}_{ii}^m(x)$ and $\bar{\xi}_{ii}^m(x)$ for the case of $\lambda_i = 1$ and $\alpha(x) = 5.5 + 4.5 \sin(2\pi x)$ (which implies that $\underline{\alpha} = 10$ and $\bar{\alpha} = 1$).

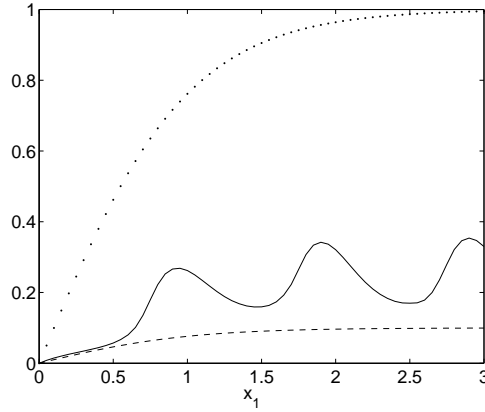


Figure 1: Graph of ξ_{ii}^m (—), $\underline{\xi}_{ii}^m$ (---) and $\bar{\xi}_{ii}^m$ (···) for the case of $\lambda_i = 1$ and $\alpha(x) = 5.5 + 4.5 \sin(2\pi x)$ (which implies that $\underline{\alpha} = 10$ and $\bar{\alpha} = 1$).

Remark 18 From (10) and (11) we obtain that

$$(13) \quad 0 \leq \underline{\alpha} \lambda_i (\underline{\xi}_{ii}^m)^2(x) < \underline{\alpha}^{-1} \quad \text{and} \quad 0 < \frac{d\underline{\xi}_{ii}^m}{dx}(x) \leq \underline{\alpha}^{-1},$$

and

$$(14) \quad 0 \leq \bar{\alpha} \lambda_i (\bar{\xi}_{ii}^m)^2(x) < \bar{\alpha}^{-1} \quad \text{and} \quad 0 < \frac{d\bar{\xi}_{ii}^m}{dx}(x) \leq \bar{\alpha}^{-1},$$

for all $x \in \mathbb{R}$, $i = 1, \dots, \infty$. Therefore,

$$(15) \quad 0 \leq \alpha(x) \lambda_i (\xi_{ii}^m)^2(x) < \alpha(x) \bar{\alpha}^{-2}$$

and

$$(16) \quad \underline{\alpha}^{-1} - \underline{\alpha} \bar{\alpha}^{-2} = \underline{\alpha}^{-1} - \underline{\alpha} \lambda_i (\bar{\xi}_{ii}^m)^2 \leq \underline{\alpha}^{-1} - \underline{\alpha} \lambda_i (\xi_{ii}^m)^2 \leq \frac{d\xi_{ii}^m}{dx}(x) \leq \bar{\alpha}^{-1},$$

for all $x \in \mathbb{R}$, $i = 1, \dots, \infty$.

4.2 Passing to the limit. Existence of solution. Regularity and Monotonicity Results

Definition 19 An operator $T \in \mathcal{L}(H, H)$ is said to be positive ($T \geq 0$) if $(Tv, v)_H \geq 0$ for all $v \in H$.

Remark 20 If $T \in \mathcal{L}(H, H)$ satisfies $T_{ij} = (Tw_i, w_j) = 0$ if $i \neq j$, then T is said to be diagonal and $T \geq 0$ if and only if $T_{ii} \geq 0$ for all $i = 1, 2, \dots$.

We can now define the solution of the Riccati equation (4) in a Hilbert-Schmidt framework.

Theorem 21 \underline{P}^m and \overline{P}^m have a limit \underline{P} and \overline{P} respectively in $\mathcal{C}([0, \infty) : \widetilde{H} \widehat{\otimes}_2 H \cap H \widehat{\otimes}_2 \widetilde{H}') \cap \mathcal{C}^1([0, \infty) : \widetilde{V} \widehat{\otimes}_2 H \cap H \widehat{\otimes}_2 \widetilde{V}')$ as $m \rightarrow \infty$. Furthermore $\underline{P}, \overline{P} \in \mathcal{C}([0, +\infty) : \mathcal{L}(V^s, V^{s+1})) \cap \mathcal{C}^1([0, +\infty) : \mathcal{L}(V^s, V^s))$, for all $s \in \mathbb{R}$ and there exist $\underline{P}_\infty, \overline{P}_\infty \in \widetilde{H} \widehat{\otimes}_2 H \cap H \widehat{\otimes}_2 \widetilde{H}'$ such that, if $0 \leq t_1 \leq t_2$, then

$$0 \leq \underline{P}(t_1) \leq \underline{P}(t_2) \leq \underline{P}_\infty$$

and

$$0 \leq \overline{P}(t_1) \leq \overline{P}(t_2) \leq \overline{P}_\infty.$$

Proof. Let us do the proof for \underline{P}^m (the other case is analogous). The proof is based on the passing to the limit of \underline{P}^m and of the differential equation satisfied by \underline{P}^m as $m \rightarrow \infty$. Now,

$$\underline{P}_\infty^m = \sum_{i=1}^m \frac{1}{\underline{\alpha} \sqrt{\lambda_i}} w_i \otimes w_i.$$

Therefore, if $0 < m_1 \leq m_2$, then

$$0 \leq \underline{P}_\infty^{m_1} \leq \underline{P}_\infty^{m_2} \quad \text{and} \quad (\underline{P}_\infty^{m_1})_{ii} = (\underline{P}_\infty^{m_2})_{ii} \quad \text{for } i = 1, \dots, m_1.$$

Further,

$$\underline{P}_\infty = \sum_{i=1}^{\infty} \frac{1}{\underline{\alpha} \sqrt{\lambda_i}} w_i \otimes w_i \in \widetilde{H} \widehat{\otimes}_2 H \cap H \widehat{\otimes}_2 \widetilde{H}',$$

since

$$\|\underline{P}_\infty\|_{\widetilde{H} \widehat{\otimes}_2 H}^2 = \|\underline{P}_\infty\|_{H \widehat{\otimes}_2 \widetilde{H}'}^2 = \frac{1}{\underline{\alpha}} \sum_{i=1}^{\infty} \frac{1}{\lambda_i^{1+\delta}} < +\infty.$$

Then, $\underline{P}_\infty^m \rightarrow \underline{P}_\infty$ in the topology of $\widetilde{H} \widehat{\otimes}_2 H \cap H \widehat{\otimes}_2 \widetilde{H}'$.

Now, since we are dealing with diagonal matrices, it is easy to prove that, if $0 < m_1 \leq m_2$, then $\underline{\xi}_{ii}^{m_1}(x) = \underline{\xi}_{ii}^{m_2}(x)$ for $i = 1, \dots, m_1$ and $0 \leq \underline{P}^{m_1}(x) \leq \underline{P}^{m_2}(x) \leq \underline{P}_\infty$, which implies that, if we define $\underline{\xi}_{ii} = \underline{\xi}_{ii}^i$, then

$$\underline{P}^m \xrightarrow{m \rightarrow \infty} \underline{P} = \sum_{i=1}^{\infty} \underline{\xi}_{ii} w_i \otimes w_i \quad \text{in } \mathcal{C}([0, \infty) : \widetilde{H} \widehat{\otimes}_2 H \cap H \widehat{\otimes}_2 \widetilde{H}'),$$

since, using (13),

$$\begin{aligned} \|\underline{P}^m - \underline{P}\|_{\mathcal{C}([0,\infty):\tilde{H}\hat{\otimes}_2 H)}^2 &= \|\underline{P}^m - \underline{P}\|_{\mathcal{C}([0,\infty):H\hat{\otimes}_2 \tilde{H}')}^2 \\ &= \sup_{x \in [0,\infty)} \sum_{i=m+1}^{\infty} \frac{|\xi_{ii}(x)|^2}{\lambda_i^\delta} \leq \underline{\alpha}^{-2} \sum_{i=m+1}^{\infty} \frac{1}{\lambda_i^{1+\delta}} \xrightarrow{m \rightarrow \infty} 0. \end{aligned}$$

Now (13) gives the expected regularity on \underline{P} because

$$\|\underline{P}\|_{\mathcal{C}([0,\infty):\mathcal{L}(V^s, V^{s+1}))}^2 = \sup_{x \in [0,\infty)} \max_i \lambda_i |\xi_{ii}(x)|^2 \leq \underline{\alpha}^{-2}.$$

Further, if $0 \leq t_1 \leq t_2$, then

$$0 \leq \underline{P}(t_1) \leq \underline{P}(t_2) \leq \underline{P}_\infty.$$

We also claim that

$$\frac{d\underline{P}^m}{dx} \xrightarrow{m \rightarrow \infty} \frac{d\underline{P}}{dx} \quad \text{in } \mathcal{C}([0, \infty) : \tilde{V}\hat{\otimes}_2 H \cap H\hat{\otimes}_2 \tilde{V}'),$$

since, from (13),

$$\begin{aligned} \left\| \frac{d\underline{P}^m}{dx} - \frac{d\underline{P}}{dx} \right\|_{\mathcal{C}([0,\infty):\tilde{V}\hat{\otimes}_2 H)}^2 &= \left\| \frac{d\underline{P}^m}{dx} - \frac{d\underline{P}}{dx} \right\|_{\mathcal{C}([0,\infty):H\hat{\otimes}_2 \tilde{V}')}^2 \\ &= \sup_{x \in [0,\infty)} \sum_{i=m+1}^{\infty} \frac{1}{\lambda_i^{1+\delta}} \left| \frac{d\xi_{ii}}{dx}(x) \right|^2 \\ &= \sup_{x \in [0,\infty)} \sum_{i=m+1}^{\infty} \frac{|\underline{\alpha}^{-1} - \underline{\alpha}\lambda_i(\xi_{ii}(x))^2|^2}{\lambda_i^{1+\delta}} \\ &\leq \sum_{i=m+1}^{\infty} \frac{\underline{\alpha}^{-2}}{\lambda_i^{1+\delta}} \xrightarrow{m \rightarrow \infty} 0. \end{aligned}$$

From (13) again

$$\left\| \frac{d\underline{P}}{dx} \right\|_{\mathcal{C}([0,+\infty):\mathcal{L}(V^s, V^s))}^2 = \sup_{x \in [0,\infty)} \max_i |\underline{\alpha}^{-1} - \underline{\alpha}\lambda_i(\xi_{ii}(x))^2|^2 \leq \underline{\alpha}^{-2}.$$

Similarly, it is easy to deduce that

$$\underline{\alpha}^{-1} I^m \xrightarrow{m \rightarrow \infty} \underline{\alpha}^{-1} I \quad \text{in } \mathcal{C}([0, \infty) : \tilde{V}\hat{\otimes}_2 H \cap H\hat{\otimes}_2 \tilde{V}')$$

and

$$\underline{\alpha} \underline{P}^m \Delta_y \underline{P}^m \xrightarrow{m \rightarrow \infty} \underline{\alpha} \underline{P} \Delta_y \underline{P} \quad \text{in } \mathcal{C}([0, \infty) : \tilde{V}\hat{\otimes}_2 H \cap H\hat{\otimes}_2 \tilde{V}').$$

Then $\underline{P} \in \mathcal{C}([0, \infty) : \tilde{H}\hat{\otimes}_2 H \cap H\hat{\otimes}_2 \tilde{H}') \cap \mathcal{C}^1([0, \infty) : \tilde{V}\hat{\otimes}_2 H \cap H\hat{\otimes}_2 \tilde{V}')$ satisfies

$$\frac{d\underline{P}}{dx} - \underline{\alpha} \underline{P} \Delta_y \underline{P} - \underline{\alpha}^{-1} I = 0,$$

where each term is considered as an element of $\tilde{V}\hat{\otimes}_2 H \cap H\hat{\otimes}_2 \tilde{V}'$. \square

Remark 22 There exists a constant $C > 0$, independent of s , such that

$$\|\underline{P}\|_{\mathcal{C}([0,+\infty):\mathcal{L}(V^s,V^{s+1}))} + \|\underline{P}\|_{\mathcal{C}^1([0,+\infty):\mathcal{L}(V^s,V^s))} \leq C$$

and

$$\|\overline{P}\|_{\mathcal{C}([0,+\infty):\mathcal{L}(V^s,V^{s+1}))} + \|\overline{P}\|_{\mathcal{C}^1([0,+\infty):\mathcal{L}(V^s,V^s))} \leq C.$$

Theorem 23 *Problem (4) has a solution P limit of P^m in $\mathcal{C}([0, a] : \widetilde{H} \widehat{\otimes}_2 H \cap H \widehat{\otimes}_2 \widetilde{H}') \cap W^{1,\infty}(0, a : \widetilde{V} \widehat{\otimes}_2 H \cap H \widehat{\otimes}_2 \widetilde{V}')$ as $m \rightarrow \infty$. Furthermore $P \in \mathcal{C}([0, a] : \mathcal{L}(V^s, V^{s+1})) \cap W^{1,\infty}(0, a : \mathcal{L}(V^s, V^s))$, for all $s \in \mathbb{R}$ and*

$$0 \leq \underline{P}(x) \leq P(x) \leq \overline{P}(x).$$

Proof. Since we are dealing with diagonal matrices, it is easy to prove that, if $0 < m_1 \leq m_2$, then $\xi_{ii}^{m_1}(x) = \xi_{ii}^{m_2}(x)$ for $i = 1, \dots, m_1$ and $0 \leq P^m(x) \leq \overline{P}_\infty$, for all $m \in \mathbb{N}$, which implies that, if we define $\xi_{ii} = \xi_{ii}^i$, then

$$P^m \xrightarrow{m \rightarrow \infty} P = \sum_{i=1}^{\infty} \xi_{ii} w_i \otimes w_i \quad \text{in } \mathcal{C}([0, a] : \widetilde{H} \widehat{\otimes}_2 H \cap H \widehat{\otimes}_2 \widetilde{H}'),$$

since, using (15),

$$\begin{aligned} \|P^m - P\|_{\mathcal{C}([0,a]:\widetilde{H}\widehat{\otimes}_2 H)}^2 &= \|P^m - P\|_{\mathcal{C}([0,a]:H\widehat{\otimes}_2 \widetilde{H}')}^2 \\ &= \sup_{x \in [0,a]} \sum_{i=m+1}^{\infty} \frac{|\xi_{ii}(x)|^2}{\lambda_i^\delta} \leq \overline{\alpha}^{-2} \sum_{i=m+1}^{\infty} \frac{1}{\lambda_i^{1+\delta}} \xrightarrow{m \rightarrow \infty} 0. \end{aligned}$$

Now (15) gives the expected regularity on P because

$$\|P\|_{\mathcal{C}([0,a]:\mathcal{L}(V^s,V^{s+1}))}^2 = \sup_{x \in [0,a]} \max_i \lambda_i |\xi_{ii}(x)|^2 \leq \overline{\alpha}^{-2}.$$

Further, if $x \in [0, a]$, then

$$0 \leq \underline{P}(x) \leq P(x) \leq \overline{P}(x) \leq P_\infty.$$

We also claim that

$$\frac{dP^m}{dx} \xrightarrow{m \rightarrow \infty} \frac{dP}{dx} \quad \text{in } L^\infty(0, a : \widetilde{V} \widehat{\otimes}_2 H \cap H \widehat{\otimes}_2 \widetilde{V}'),$$

since, from (16),

$$\begin{aligned} \left\| \frac{dP^m}{dx} - \frac{dP}{dx} \right\|_{L^\infty(0,a:\widetilde{V}\widehat{\otimes}_2 H)}^2 &= \left\| \frac{dP^m}{dx} - \frac{dP}{dx} \right\|_{L^\infty(0,a:H\widehat{\otimes}_2 \widetilde{V}')}^2 \\ &= \text{ess sup}_{x \in (0,a)} \sum_{i=m+1}^{\infty} \frac{1}{\lambda_i^{1+\delta}} \left| \frac{d\xi_{ii}}{dx}(x) \right|^2 \end{aligned}$$

$$\begin{aligned}
&= \operatorname{ess\,sup}_{x \in (0,a)} \sum_{i=m+1}^{\infty} \frac{|\alpha(x)^{-1} - \alpha(x)\lambda_i(\xi_{ii}(x))^2|^2}{\lambda_i^{1+\delta}} \\
&\leq \sum_{i=m+1}^{\infty} \max \left\{ \bar{\alpha}^{-2}, \left(\underline{\alpha} \lambda_i \bar{\xi}_{ii,\infty}^m - \underline{\alpha}^{-1} \right)^2 \right\} \frac{1}{\lambda_i^{1+\delta}} \\
&\leq \max \left\{ \bar{\alpha}^{-2}, \lambda_1 \underline{\alpha}^2 \bar{\alpha}^{-2} \right\} \sum_{i=m+1}^{\infty} \frac{1}{\lambda_i^{1+\delta}} \xrightarrow{m \rightarrow \infty} 0.
\end{aligned}$$

From (16) again

$$\begin{aligned}
\left\| \frac{dP}{dx} \right\|_{L^\infty(0,a;\mathcal{L}(V^s,V^s))}^2 &= \operatorname{ess\,sup}_{x \in (0,a)} \max_i |\alpha(x)^{-1} - \alpha(x)\lambda_i(\xi_{ii}(x))^2|^2 \\
&\leq \max \left\{ \bar{\alpha}^{-2}, \lambda_1 \underline{\alpha}^2 \bar{\alpha}^{-2} \right\}.
\end{aligned}$$

Similarly, it is easy to deduce that

$$\alpha^{-1} I^m \xrightarrow{m \rightarrow \infty} \alpha^{-1} I \quad \text{in } L^\infty(0,a : \tilde{V} \hat{\otimes}_2 H \cap H \hat{\otimes}_2 \tilde{V}')$$

and

$$\alpha P^m \Delta_y P^m \xrightarrow{m \rightarrow \infty} \alpha P \Delta_y P \quad \text{in } L^\infty(0,a : \tilde{V} \hat{\otimes}_2 H \cap H \hat{\otimes}_2 \tilde{V}').$$

Then $P \in \mathcal{C}([0,a] : \tilde{H} \hat{\otimes}_2 H \cap H \hat{\otimes}_2 \tilde{H}') \cap W^{1,\infty}(0,a : \tilde{V} \hat{\otimes}_2 H \cap H \hat{\otimes}_2 \tilde{V}')$ satisfies

$$\frac{dP}{dx} - \alpha P \Delta_y P - \alpha^{-1} I = 0,$$

where each term is considered as an element of $\tilde{V} \hat{\otimes}_2 H \cap H \hat{\otimes}_2 \tilde{V}'$. \square

Remark 24 There exists a constant $C > 0$, independent of s , such that

$$\|P\|_{\mathcal{C}([0,a];\mathcal{L}(V^s,V^{s+1}))} + \|P\|_{W^{1,\infty}(0,a;\mathcal{L}(V^s,V^s))} \leq C.$$

Remark 25 If $\alpha \in \mathcal{C}([0,a])$ then

$$P^m \xrightarrow{m \rightarrow \infty} P \text{ in } \mathcal{C}([0,a] : \tilde{H} \hat{\otimes}_2 H \cap H \hat{\otimes}_2 \tilde{H}') \cap \mathcal{C}^1([0,a] : \tilde{V} \hat{\otimes}_2 H \cap H \hat{\otimes}_2 \tilde{V}')$$

and $P \in \mathcal{C}([0,a] : \mathcal{L}(V^s,V^{s+1})) \cap \mathcal{C}^1([0,a] : \mathcal{L}(V^s,V^s))$, for all $s \in \mathbb{R}$. Furthermore, there exists a constant $C > 0$, independent of s , such that

$$\|P\|_{\mathcal{C}([0,a];\mathcal{L}(V^s,V^{s+1}))} + \|P\|_{\mathcal{C}^1([0,a];\mathcal{L}(V^s,V^s))} \leq C.$$

4.3 Uniqueness of Solution

Lemma 26 Suppose T is a symmetric operator satisfying $T \in \mathcal{L}(V^p, V^{p+r})$ for some $r \in \mathbb{R}$ and for all $p \geq k$ ($k \in \mathbb{R}$) and there exists a constant $C > 0$ independent of p such that $\|T\|_{\mathcal{L}(V^p, V^{p+r})}^2 \leq C$, then T is diagonal, i.e., $T_{i,j} = (Tw_i, w_j)_H = 0$ if $i \neq j$.

Proof.

$$\begin{aligned} \|T\|_{\mathcal{L}(V^p, V^{p+r})}^2 &\geq \sup_i \|T \frac{w_i}{\|w_i\|_{V^p}}\|_{V^{p+r}}^2 = \sup_i \|\lambda_i^{-p/2} Tw_i\|_{V^{p+r}}^2 \\ &= \sup_i \|\lambda_i^{-p/2} \sum_{j=1}^{\infty} T_{i,j} w_j\|_{V^{p+r}}^2 = \sup_i \lambda_i^{-p} \sum_{j=1}^{\infty} \lambda_j^{p+r} |T_{i,j}|^2 \end{aligned}$$

Therefore, $T_{i,j} = 0$ if $i \neq j$ and $\|T\|_{\mathcal{L}(V^p, V^{p+r})}^2 \geq \sup_i \lambda_i^r |T_{i,i}|^2$, because, otherwise, since

$$\lambda_i^{-p} \lambda_j^p \rightarrow \infty \text{ as } p \rightarrow \infty \text{ (if } \lambda_j > \lambda_i),$$

we would have

$$\|T\|_{\mathcal{L}(V^p, V^{p+r})}^2 \rightarrow \infty \text{ as } p \rightarrow \infty$$

in contradiction with the assumptions. \blacksquare

Theorem 27 The solution P of (4) is unique in the set of functions

$$\{P \in W^{1,\infty}(0, a; V^p \widehat{\otimes}_2 V^q) \text{ for some } p, q \in \mathbb{R} :$$

$$P \in \mathcal{C}([0, +\infty) : \mathcal{L}(V^s, V^{s+1})) \text{ for all } s \in \mathbb{R},$$

$$\text{with } \|P\|_{\mathcal{C}([0, +\infty) : \mathcal{L}(V^s, V^{s+1}))}^2 \text{ uniformly bounded in } s\}.$$

Proof. Let P a solution of (4) in the above class of functions. From Theorem 26 P is diagonal. Let

$$P(x) = \sum_{i=1}^{\infty} \xi_{ii}(x) w_i \otimes w_i \in W^{1,\infty}(0, a; V^p \widehat{\otimes}_2 V^q)$$

be the Fourier expansion of $P(x)$. Then, equation (4) can be written as

$$\sum_{i=1}^{\infty} \left(\frac{d\xi_{ii}}{dx}(x) + \alpha(x) \lambda_i \xi_{ii}^2 \right) w_i \otimes w_i = \sum_{i=1}^{\infty} \alpha(x)^{-1} w_i \otimes w_i,$$

with both terms of the above equality in $(\widetilde{V} \widehat{\otimes}_2 H) \cap (H \widehat{\otimes}_2 \widetilde{V}')$. Then, by the uniqueness of the Fourier representation, we have that

$$\frac{d\xi_{ii}}{dx}(x) + \alpha(x) \lambda_i \xi_{ii}^2 = \alpha(x)^{-1}.$$

Therefore, if ξ^m is the matrix given by $\xi^m = (\xi_{ij})_{i,j=1}^m$, then ξ^m is the unique solution of (6) (as we proved in Section 4.1.1), which concludes the proof due to the strong convergence of ξ^m to ξ in $\mathcal{C}([0, a] : \widetilde{H} \widehat{\otimes}_2 H \cap H \widehat{\otimes}_2 \widetilde{H}')$ proved in Theorem 23. \square

5 Elliptic problems associated to the Riccati Equation (4)

In this section we show how the Hilbert-Schmidt operator P solution of the Riccati equation (4) can be used to transform the second order elliptic boundary value problem (1) into one or two uncoupled first order initial value problems (system (2)).

5.1 Function r

Theorem 28 *Let $s \in \mathbb{R}$ arbitrary and $u_0 \in V^s$. Then, there exists a unique solution $r \in \mathcal{C}([0, a] : V^s) \cap L^2(0, a : V^{s+1/2}) \cap H^1(0, a : V^{s-1/2})$, of the initial value problem*

$$\begin{cases} \frac{dr}{dx} - \alpha P \Delta_y r = 0, \\ r(0) = u_0. \end{cases}$$

Proof. Let

$$u_0 = \sum_{i=1}^{\infty} u_{0i} w_i$$

be the Fourier decomposition of u_0 . Then, the solution of the problem is given by

$$r(x, y) = \sum_{i=1}^{\infty} r_i(x) w_i(y),$$

where each Fourier coefficient $r_i(x)$, $i = 1, \dots, \infty$, satisfies the differential equation

$$\begin{cases} \frac{dr_i}{dx} = -\alpha \lambda_i \xi_{ii} r_i, \\ r_i(0) = u_{0i}. \end{cases}$$

Then,

$$r_i(x) = u_{0i} e^{-\int_0^x \lambda_i \alpha(t) \xi_{ii}(t) dt} \in \mathcal{C}([0, a])$$

and

$$\begin{aligned} |r_i(x)| &\leq |u_{0i}| e^{-\bar{\alpha} \lambda_i \int_0^x \xi_{ii}(t) dt} = |u_{0i}| e^{-\frac{\bar{\alpha} \lambda_i}{\alpha \sqrt{\lambda_i}} \int_0^x \frac{\sinh(\sqrt{\lambda_i} t)}{\cosh(\sqrt{\lambda_i} t)} dt} \\ &= \frac{|u_{0i}|}{(\cosh \sqrt{\lambda_i} x)^{\bar{\alpha} \alpha^{-1}}} \leq |u_{0i}|. \end{aligned}$$

Now, for all $x \in [0, a]$, we have that

$$\sum_{i=1}^{\infty} \lambda_i^s |r_i(x)|^2 \leq \sum_{i=1}^{\infty} \lambda_i^s |u_{0i}|^2 < \infty,$$

which implies that $r(x) \in V^s$. Hence, since $r_i \in \mathcal{C}([0, a])$, $i = 1, \dots, \infty$, and thanks to the uniform convergence of the previous series, we have that $r \in \mathcal{C}([0, a] : V^s)$. To obtain the $L^2(0, a : V^{s+1/2})$ regularity consider

$$\begin{aligned} \int_0^a \sum_{i=1}^{\infty} \lambda_i^{s+1/2} |r_i(x)|^2 dx &\leq 2^{2\bar{\alpha}\underline{\alpha}^{-1}} \sum_{i=1}^{\infty} \lambda_i^s |u_{0i}|^2 \int_0^a \frac{\sqrt{\lambda_i}}{e^{2\bar{\alpha}\underline{\alpha}^{-1}\sqrt{\lambda_i}x}} dx \\ &= \frac{2^{2\bar{\alpha}\underline{\alpha}^{-1}}}{2\bar{\alpha}\underline{\alpha}^{-1}} \sum_{i=1}^{\infty} (1 - e^{-2\bar{\alpha}\underline{\alpha}^{-1}\sqrt{\lambda_i}a}) \lambda_i^s |u_{0i}|^2 < \infty. \end{aligned}$$

To obtain the $H^1(0, a : V^{s-1/2})$ regularity consider

$$\begin{aligned} \int_0^a \sum_{i=1}^{\infty} \lambda_i^{s-1/2} \left| \frac{dr_i}{dx}(x) \right|^2 dx &\leq 2^{2\bar{\alpha}\underline{\alpha}^{-1}} \underline{\alpha}^2 \bar{\alpha}^{-2} \sum_{i=1}^{\infty} \lambda_i^s |u_{0i}|^2 \int_0^a \frac{\sqrt{\lambda_i}}{e^{2\bar{\alpha}\underline{\alpha}^{-1}\sqrt{\lambda_i}x}} dx \\ &= 2^{(2\bar{\alpha}\underline{\alpha}^{-1}-1)} \underline{\alpha}^3 \bar{\alpha}^{-3} \sum_{i=1}^{\infty} (1 - e^{-2\bar{\alpha}\underline{\alpha}^{-1}\sqrt{\lambda_i}a}) \lambda_i^s |u_{0i}|^2 < \infty. \quad \square \end{aligned}$$

Remark 29 In order to prove that $r \in \mathcal{C}([0, a] : V^s)$ we could have also used the theory of intermediate spaces (see, for instance, [10]), which gives

$$L^2(0, a : V^{s+1/2}) \cap H^1(0, a : V^{s-1/2}) \subset \mathcal{C}([0, a] : [V^{s-1/2}, V^{s+1/2}]_{1/2})$$

and $[V^{s-1/2}, V^{s+1/2}]_{1/2} = V^s$.

Proposition 30 *If $\alpha \in \mathcal{C}([0, a])$ then $r \in \mathcal{C}^1([0, a] : V^{s-1}) \cap \mathcal{C}^1([0, a] : V^p)$, for any $p \in \mathbb{N}$.*

Proof. Let us prove now that $r \in \mathcal{C}^1([0, a] : V^{s-1})$. We have

$$\left| \frac{dr_i}{dx}(x) \right| \leq \underline{\alpha} \bar{\alpha}^{-1} \sqrt{\lambda_i} \frac{\sinh(\sqrt{\lambda_i}x)}{(\cosh(\sqrt{\lambda_i}x))^{1+\bar{\alpha}\underline{\alpha}^{-1}}} |u_{0i}| \leq \underline{\alpha} \bar{\alpha}^{-1} \sqrt{\lambda_i} |u_{0i}|$$

which implies that, for all $x \in [0, a]$,

$$\sum_{i=1}^{\infty} \lambda_i^{s-1} \left| \frac{dr_i}{dx}(x) \right|^2 \leq \underline{\alpha}^2 \bar{\alpha}^{-2} \sum_{i=1}^{\infty} \lambda_i^s |u_{0i}|^2 < \infty.$$

Thus, $\frac{dr}{dx}(x, \cdot) \in V^{s-1}$, for all $x \in [0, a]$, and therefore $r \in \mathcal{C}^1([0, a] : V^{s-1})$.

To prove the additional regularity we consider

$$\sum_{i=1}^{\infty} \lambda_i^p |r_i(x)|^2 \leq 2^{2\bar{\alpha}\underline{\alpha}^{-1}} \sum_{i=1}^{\infty} \lambda_i^s |u_{0i}|^2 \lambda_i^{p-s} e^{-2\bar{\alpha}\underline{\alpha}^{-1}\sqrt{\lambda_i}x} < \infty$$

for any $x > 0$, since for any $\mu, k > 0$

$$\lambda^k e^{-\mu\sqrt{\lambda}} \longrightarrow 0 \quad \text{as } \lambda \rightarrow +\infty.$$

Similarly, it can be proved that $\frac{dr}{dx} \in \mathcal{C}([0, a] : V^p)$. \square

Theorem 31 *Let $s \in \mathbb{R}$ arbitrary and $f \in L^2(0, a : V^s)$. Then, there exists a unique solution $r \in \mathcal{C}([0, a] : V^{s+3/2}) \cap L^2(0, a : V^{s+2}) \cap H^1(0, a : V^{s+1})$, of the initial value problem*

$$\begin{cases} \frac{dr}{dx} - \alpha P \Delta_y r = P f, \\ r(0) = 0. \end{cases}$$

Proof. Let

$$f(x, \cdot) = \sum_{i=1}^{\infty} f_i(x) w_i$$

be the Fourier decomposition of $f(x, \cdot)$, for all $x \in [0, a]$ (respectively, a.e. in $(0, a)$), with $f_i \in \mathcal{C}([0, a])$ (respectively, $f_i \in L^2(0, a)$). Then, the solution of the problem is given by

$$r(x, y) = \sum_{i=1}^{\infty} r_i(x) w_i(y),$$

where each Fourier coefficient $r_i(x)$, $i = 1, \dots, \infty$, satisfies the differential equation

$$(17) \quad \begin{cases} \frac{dr_i}{dx} = \xi_{ii} (-\alpha \lambda_i r_i + f_i), \\ r_i(0) = 0. \end{cases}$$

Then, multiplying in (17) by r_i and integrating over $(0, x)$ we obtain

$$\frac{1}{2} |r_i(x)|^2 + \int_0^x \alpha(s) \lambda_i \xi_{ii}(s) |r_i(s)|^2 ds = \int_0^x \xi_{ii}(s) f_i(s) r_i(s) ds.$$

Then

$$\begin{aligned} & \frac{1}{2} |r_i(x)|^2 + \int_0^x \alpha(s) \lambda_i \xi_{ii}(s) |r_i(s)|^2 ds \\ & \leq \frac{1}{2} \int_0^x \bar{\alpha} \lambda_i \xi_{ii}(s) |r_i(s)|^2 ds + \frac{1}{2} \int_0^x \frac{1}{\bar{\alpha} \lambda_i} \xi_{ii}(s) |f_i(s)|^2 ds \end{aligned}$$

and therefore, we obtain

$$|r_i(x)|^2 \leq \int_0^x \frac{1}{\bar{\alpha}^2 \lambda_i^{1+1/2}} |f_i(s)|^2 ds.$$

Thus,

$$\sum_{i=1}^{\infty} \lambda_i^{s+3/2} |r_i(x)|^2 \leq \frac{1}{\bar{\alpha}^2} \sum_{i=1}^{\infty} \lambda_i^s \|f_i\|_{L^2(0, a)}^2 = \frac{1}{\bar{\alpha}^2} \|f\|_{L^2(0, a : V^s)}^2 < \infty,$$

which implies that $r(x) \in V^{s+3/2}$. Hence, since $r_i \in \mathcal{C}([0, a])$ (because $f_i \in L^1(0, a)$), $i = 1, \dots, \infty$, and thanks to the uniform convergence of the series we have that $r \in \mathcal{C}([0, a] : V^{s+3/2})$.

To obtain the $L^2(0, a : V^{s+2})$ regularity consider

$$\begin{aligned} \int_0^a |r_i(x)|^2 dx &\leq \int_0^a \int_0^x \frac{1}{\bar{\alpha} \lambda_i} \xi_{ii}(s) |f_i(s)|^2 ds dx \leq \int_0^a \frac{1}{\bar{\alpha} \lambda_i} \bar{\xi}_{ii}(x) \|f_i\|_{L^2(0,a)}^2 dx \\ &= \|f_i\|_{L^2(0,a)}^2 \frac{1}{\bar{\alpha}^2 \lambda_i^2} \ln(\cosh a). \end{aligned}$$

Therefore

$$\begin{aligned} \|r\|_{L^2(0,a;V^{s+2})}^2 &= \int_0^a \sum_{i=1}^{\infty} \lambda_i^{s+2} |r_i(x)|^2 dx \leq \frac{\ln(\cosh a)}{\bar{\alpha}^2} \sum_{i=1}^{\infty} \lambda_i^s \|f_i\|_{L^2(0,a)}^2 \\ &= \frac{\ln(\cosh a)}{\bar{\alpha}^2} \|f\|_{L^2(0,a;V^s)}^2 < \infty. \end{aligned}$$

Finally, since $\xi_{ii} \in L^\infty(0, a)$ is uniformly bounded and $f_i \in L^2(0, a)$, from (17) and Theorem 23 we have that $r_i \in H^1(0, a)$ and we deduce that $r \in H^1(0, a : V^{s+1})$. \square

Remark 32 The solution r_i of (17) is

$$(18) \quad r_i(x) = e^{-\int_0^x \alpha(t) \lambda_i \xi_{ii}(t) dt} \int_0^x \xi_{ii}(s) f_i(s) e^{\int_0^s \alpha(t) \lambda_i \xi_{ii}(t) dt} ds.$$

Remark 33 In order to prove that $r \in \mathcal{C}([0, a] : V^{s+3/2})$ we could have also used the theory of intermediate spaces (see, for instance, [10]), which gives

$$L^2(0, a : V^{s+2}) \cap H^1(0, a : V^{s+1}) \subset \mathcal{C}([0, a] : [V^{s+1}, V^{s+2}]_{1/2})$$

and $[V^{s+1}, V^{s+2}]_{1/2} = V^{s+3/2}$.

Remark 34 If $f \in L^\infty(0, a : V^s)$ then $r \in W^{1,\infty}(0, a : V^{s+1/2})$. Furthermore, if $\alpha \in \mathcal{C}([0, a])$ and $f \in \mathcal{C}([0, a] : V^s)$ then $r \in \mathcal{C}^1([0, a] : V^{s+1/2})$.

Corollary 35 Let $j, k \in \mathbb{R}$ arbitrary, $u_0 \in V^j$ and $f \in L^2(0, a : V^k)$. Then, if $p = \min\{j, k + 3/2\}$, there exists a unique solution $r \in \mathcal{C}([0, a] : V^p) \cap L^2(0, a : V^{p+1/2}) \cap H^1(0, a : V^{p-1/2})$, of the initial value problem

$$(19) \quad \begin{cases} \frac{dr}{dx} - \alpha P \Delta_y r = P f, \\ r(0) = u_0. \end{cases}$$

Furthermore, each Fourier coefficient r_i , $i = 1, \dots, \infty$, is given by

$$r_i(x) = e^{-\int_0^x \alpha(t) \lambda_i \xi_{ii}(t) dt} \left(u_{0i} + \int_0^x \xi_{ii}(s) f_i(s) e^{\int_0^s \alpha(t) \lambda_i \xi_{ii}(t) dt} ds \right).$$

Remark 36 With the assumptions of the above Corollary, if $\alpha \in \mathcal{C}([0, a])$ and $f \in \mathcal{C}([0, a] : V^k)$, then $r \in \mathcal{C}^1([0, a] : V^{p-1})$.

5.2 Dirichlet Condition on Γ_a

In the following Lemma 37 and Corollary 38 we give existence and uniqueness results for the corresponding solution u in a suitable functional space which enables us to prove that u is solution of a Poisson equation. Therefore we do not try to obtain all and/or the best regularity results, since they can be obtained later as a result of the well-known regularity results for the Poisson equation.

Lemma 37 *Let $j, k \in \mathbb{R}$ arbitrary, $u_0 \in V^j$ and $f \in L^2(0, a : V^k)$. Then, if $p = \min\{j, k + 3/2\}$, there exists a unique solution $u \in \mathcal{C}([0, a] : V^p)$ of the initial value problem*

$$(20) \quad \begin{cases} P\alpha \frac{du}{dx} = u - r, \\ u(a) = 0, \end{cases}$$

where $r \in \mathcal{C}([0, a] : V^p) \cap L^2(0, a : V^{p+1/2}) \cap H^1(0, a : V^{p-1/2})$ is the solution of the initial value problem (19). Furthermore, $u = \sum_{i=1}^{\infty} u_i w_i$, with $u_i \in \mathcal{C}([0, a])$, $\frac{du_i}{dx} \in L^\infty(0, a)$ and $\alpha \frac{du_i}{dx} \in W^{1,\infty}(0, a)$, for all $i \in \mathbb{N}$.

Proof. The solution of the problem is given by

$$u(x, y) = \sum_{i=1}^{\infty} u_i(x) w_i(y),$$

where each Fourier coefficient $u_i(x)$, $i = 1, \dots, \infty$, satisfies the differential equation

$$(21) \quad \begin{cases} \xi_{ii} \alpha \frac{du_i}{dx} = u_i - r_i, \\ u_i(a) = 0. \end{cases}$$

Solving (21) we obtain

$$(22) \quad \begin{aligned} u_i(x) &= e^{-\int_x^a (\alpha(t) \xi_{ii}(t))^{-1} dt} \int_x^a (\alpha(s) \xi_{ii}(s))^{-1} r_i(s) e^{\int_s^a (\alpha(t) \xi_{ii}(t))^{-1} dt} ds \\ &= e^{-\int_x^a (\alpha(t) \xi_{ii}(t))^{-1} dt} \left(r_i(x) e^{\int_x^a (\alpha(t) \xi_{ii}(t))^{-1} dt} + \int_x^a \left(\frac{dr_i}{dx}(s) \right) e^{\int_s^a (\alpha(t) \xi_{ii}(t))^{-1} dt} ds \right) \\ &= r_i(x) + e^{-\int_x^a (\alpha(t) \xi_{ii}(t))^{-1} dt} \int_x^a \left(\frac{dr_i}{dx}(s) \right) e^{\int_s^a (\alpha(t) \xi_{ii}(t))^{-1} dt} ds. \end{aligned}$$

Now ξ_{ii} is 0 at $x = 0$ and we have to check the behaviour of u_i near $x = 0$. Let us see that $u_i \in \mathcal{C}([0, a])$, $i = 1, \dots, \infty$. It is easy to see that

$$e^{-\int_x^a (\alpha(t) \xi_{ii}(t))^{-1} dt} \leq \left(\frac{\sinh(\sqrt{\lambda_i} x)}{\sinh(\sqrt{\lambda_i} a)} \right)^{\bar{\alpha}\alpha^{-1}} \rightarrow 0 \quad \text{as } x \rightarrow 0.$$

Now, since $r_i \in \mathcal{C}([0, a])$, it is easy to prove that, if $r_i(0) = u_{0i} \neq 0$, then

$$(23) \quad \lim_{x \rightarrow 0} \left| \int_x^a (\alpha(s) \xi_{ii}(s))^{-1} r_i(s) e^{\int_s^a (\alpha(t) \xi_{ii}(t))^{-1} dt} ds \right| = \infty.$$

Therefore, if $u_{0i} = 0$ and (23) does not hold, then (obviously)

$$\lim_{x \rightarrow 0} u_i(x) = 0.$$

Otherwise, using L'Hôpital's rule,

$$\begin{aligned} \lim_{x \rightarrow 0} u_i(x) &= \lim_{x \rightarrow 0} \frac{\int_x^a (\alpha(s) \xi_{ii}(s))^{-1} r_i(s) e^{\int_s^a (\alpha(t) \xi_{ii}(t))^{-1} dt} ds}{e^{\int_x^a (\alpha(t) \xi_{ii}(t))^{-1} dt}} \\ &= \lim_{x \rightarrow 0} \frac{(\alpha(x) \xi_{ii}(x))^{-1} r_i(x) e^{\int_x^a (\alpha(t) \xi_{ii}(t))^{-1} dt}}{(\alpha(x) \xi_{ii}(x))^{-1} e^{\int_x^a (\alpha(t) \xi_{ii}(t))^{-1} dt}} = r_i(0) = u_{0i}. \end{aligned}$$

This implies that $u_i \in \mathcal{C}([0, a])$ for all $i = 1, \dots, \infty$. The results about $\frac{du_i}{dx}$ and $\alpha \frac{du_i}{dx}$ are easily verified.

Let us prove that $u \in \mathcal{C}([0, a] : V^p)$. From (21) we have

$$\begin{aligned} &\int_x^a \xi_{ii}(s) \alpha(s) \left(\frac{du_i}{dx}(s) - \frac{dr_i}{dx}(s) \right)^2 ds \\ &= \frac{1}{2} \int_x^a \frac{d}{dx} (u_i(s) - r_i(s))^2 ds - \int_x^a \xi_{ii}(s) \alpha(s) \frac{dr_i}{dx}(s) \left(\frac{du_i}{dx}(s) - \frac{dr_i}{dx}(s) \right) ds \\ &= -\frac{1}{2} (u_i(x) - r_i(x))^2 + \frac{1}{2} |r_i(a)|^2 - \int_x^a \xi_{ii}(s) \alpha(s) \frac{dr_i}{dx}(s) \left(\frac{du_i}{dx}(s) - \frac{dr_i}{dx}(s) \right) ds. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{1}{2} (u_i(x) - r_i(x))^2 + \int_x^a \xi_{ii}(s) \alpha(s) \left(\frac{du_i}{dx}(s) - \frac{dr_i}{dx}(s) \right)^2 ds \leq \\ &\frac{1}{2} |r_i(a)|^2 + \frac{1}{2} \int_x^a \xi_{ii}(s) \alpha(s) \left| \frac{dr_i}{dx}(s) \right|^2 ds + \frac{1}{2} \int_x^a \xi_{ii}(s) \alpha(s) \left(\frac{du_i}{dx}(s) - \frac{dr_i}{dx}(s) \right)^2 ds. \end{aligned}$$

Hence,

$$\begin{aligned} &|u_i(x)|^2 + |r_i(x)|^2 + \int_x^a \xi_{ii}(s) \alpha(s) \left(\frac{du_i}{dx}(s) - \frac{dr_i}{dx}(s) \right)^2 ds \leq \\ &\frac{|u_i(x)|^2}{2} + 2|r_i(x)|^2 + |r_i(a)|^2 + \frac{\alpha}{\alpha} \int_0^a \frac{1}{\sqrt{\lambda_i}} \left| \frac{dr_i}{dx}(s) \right|^2 ds, \end{aligned}$$

which implies that

$$\sum_{i=1}^{\infty} \lambda_i^p |u_i(x)|^2 \leq 2 \sum_{i=1}^{\infty} \lambda_i^p (|r_i(x)|^2 + |r_i(a)|^2) + 2 \frac{\alpha}{\alpha} \left\| \frac{dr}{dx} \right\|_{H^1(0,a;V^{p-1/2})} < \infty.$$

Therefore $u(x, \cdot) \in V^p$ for all $x \in [0, a]$ and thanks to the uniform convergence of the series we have that $u \in \mathcal{C}([0, a] : V^p)$, since $u_i \in \mathcal{C}([0, a])$ and

$$\begin{aligned} \|u - \sum_{i=1}^m u_i w_i\|_{\mathcal{C}([0,a]:V^p)} &= \sup_{x \in [0,a]} \sum_{i=m+1}^{\infty} \lambda_i^p |u_i(x)|^2 \\ &\leq 2 \sum_{i=m+1}^{\infty} \lambda_i^p |r_i(a)|^2 + \frac{\alpha}{\alpha} \sum_{i=m+1}^{\infty} \lambda_i^{p-1/2} \int_0^a \left| \frac{dr_i}{dx}(s) \right|^2 ds \xrightarrow{m \rightarrow \infty} 0. \quad \square \end{aligned}$$

Corollary 38 *Let $j, k, q \in \mathbb{R}$ arbitrary, $u_0 \in V^j$, $f \in L^2(0, a : V^k)$ and $u_a \in V^q$. Then, if $p = \min\{j, k + 3/2, q\}$, there exists a unique solution $u \in \mathcal{C}([0, a] : V^p)$ of the initial value problem*

$$(24) \quad \begin{cases} P\alpha \frac{du}{dx} = u - r, \\ u(a) = u_a, \end{cases}$$

where $r \in \mathcal{C}([0, a] : V^{\bar{p}}) \cap L^2(0, a : V^{\bar{p}+1/2}) \cap H^1(0, a : V^{\bar{p}-1/2})$, is the solution of the initial value problem (19), with $\bar{p} = \min\{j, k + 3/2\}$. Furthermore, each Fourier coefficient u_i , $i = 1, \dots, \infty$, satisfies $u_i \in \mathcal{C}([0, a])$, $\frac{du_i}{dx} \in L^\infty(0, a)$, $\alpha \frac{du_i}{dx} \in W^{1,\infty}(0, a)$ and

$$(25) \quad \begin{aligned} u_i(x) &= e^{-\int_x^a (\alpha(t)\xi_{ii}(t))^{-1} dt} \\ &\quad \left(u_{ai} + \int_x^a (\alpha(s)\xi_{ii}(s))^{-1} r_i(s) e^{\int_s^a (\alpha(t)\xi_{ii}(t))^{-1} dt} ds \right). \end{aligned}$$

Theorem 39 *Let $j, k, q \in \mathbb{R}$ arbitrary, $u_0 \in V^j$, $f \in L^2(0, a : V^k)$ and $u_a \in V^q$. Then, if $p = \min\{j, k + 3/2, q\}$, the unique solution $u \in \mathcal{C}([0, a] : V^p)$ of the initial value problem (24) is also the unique solution of the elliptic problem*

$$(26) \quad \begin{cases} -\nabla \cdot (\alpha(x)\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Sigma, \\ u = u_0 & \text{on } \Gamma_0, \\ u = u_a & \text{on } \Gamma_a. \end{cases}$$

Proof. Derivating (21) with respect to the x variable we get

$$\frac{du_i}{dx} - \frac{d\xi_{ii}}{dx} \alpha \frac{du_i}{dx} - \frac{dr_i}{dx} = \xi_{ii} \frac{d}{dx} \left(\alpha \frac{du_i}{dx} \right),$$

which is an equation with all its terms in $L^\infty(0, a)$ (see Theorem 23 and regularity results of Corollary 38). Then, taking into account the Riccati equation satisfied by ξ_{ii} and the equation satisfied by r_i , we have that

$$\xi_{ii} \frac{d}{dx} \left(\alpha \frac{du_i}{dx} \right) = \alpha^2 \lambda_i \xi_{ii}^2 \frac{du_i}{dx} + \xi_{ii} (\alpha \lambda_i r_i - f_i).$$

Therefore, we have the following equation in $L^2(0, a)$:

$$\xi_{ii} \left[\frac{d}{dx} \left(\alpha \frac{du_i}{dx} \right) \right] = \xi_{ii} [\alpha \lambda_i u_i - f_i].$$

Hence, since $\xi_{ii}(x) > 0$ for all $x > 0$, we have that

$$\frac{d}{dx} \left(\alpha \frac{du}{dx} \right) = -\alpha \Delta_y u - f \quad \text{in } L^2(0, a : V^{p-3}),$$

which implies that

$$-\nabla \cdot (\alpha(x) \nabla u) = -\frac{d}{dx} \left(\alpha \frac{du}{dx} \right) - \alpha \Delta_y u = f \quad \text{in } L^2(0, a : V^k).$$

Finally, the boundary conditions are obviously satisfied, which concludes the proof. \square

Example 40 If $f \in L^2(0, a : H^{-1}(\mathcal{O}))$ and $u_0, u_a \in V^{1/2} = H_{00}^{1/2}(\mathcal{O})$, then $u \in \mathcal{C}([0, a] : H_{00}^{1/2}(\Omega))$, since $H^{-1}(\Omega) = V^{-1}$ and $H_{00}^{1/2}(\Omega) = V^{1/2}$.

Remark 41 Once we know that the solution of (24) is the unique solution of (26) we can improve the regularity results showed in Lemma 37 and Corollary 38. Some of the regularity results for these kind of problems can be seen, for instance, in [6].

Remark 42 The above explicit formula (25) is not well-suited for computational purposes, since at $x = 0$ there is a multiplication $0 \cdot \infty$. Nevertheless, we can avoid this problem by developing the right hand term, obtaining the following result

Theorem 43 *Each Fourier coefficient u_i , $i = 1, \dots, \infty$, of the solution u of the initial value problem (24) (or, equivalently, problem (26)), is given by*

$$\begin{aligned} u_i(x) &= r_i(x) - r_i(a) e^{-\int_x^a (\alpha(t) \xi_{ii}(t))^{-1} dt} \\ &\quad + e^{-\int_x^a (\alpha(t) \xi_{ii}(t))^{-1} dt} \int_x^a e^{\int_s^a \lambda_i \alpha(t) \xi_{ii}(t) dt} (-\alpha(s) \lambda_i r_i(s) + f_i(s)) ds. \end{aligned}$$

Proof. Let us suppose that $u_a \equiv 0$ (otherwise we just have to add the additional term given by the last term of the formula (25)). Integrating by parts in (25), for $i = 1, \dots, \infty$, we obtain

$$\begin{aligned} u_i(x) &= r_i(x) - r_i(a) e^{-\int_x^a (\alpha(t) \xi_{ii}(t))^{-1} dt} \\ &\quad + e^{-\int_x^a (\alpha(t) \xi_{ii}(t))^{-1} dt} \int_x^a e^{\int_s^a (\alpha(t) \xi_{ii}(t))^{-1} dt} \frac{dr_i}{dx}(s) ds \\ &= r_i(x) - r_i(a) e^{-\int_x^a (\alpha(t) \xi_{ii}(t))^{-1} dt} \\ &\quad + e^{-\int_x^a (\alpha(t) \xi_{ii}(t))^{-1} dt} \int_x^a e^{\int_s^a \lambda_i \alpha(t) \xi_{ii}(t) dt} (-\alpha(s) \lambda_i r_i(s) + f_i(s)) ds, \end{aligned}$$

since

$$v = \xi_{ii} e^{\int_s^a (\alpha(t) \xi_{ii}(t))^{-1} dt}$$

satisfies

$$\frac{dv}{dx} = -v\alpha\lambda_i\xi$$

and therefore,

$$v(x) = e^{\int_s^a \lambda_i \alpha(t) \xi_{ii}(t) dt}. \quad \square$$

Example 44 Let $f_i(x) \equiv f_i$, constant in \mathcal{O} and $\alpha \equiv 1$ in $(0, a)$. By Theorem 43 we have that the solution of problem (24) (or, equivalently, problem (26)) is

$$\begin{aligned} u_i(x) = & f_i \frac{\sinh(\sqrt{\lambda_i}x)}{\lambda_i} \left(e^{-\sqrt{\lambda_i}a} - e^{-\sqrt{\lambda_i}a} \frac{\sinh(\sqrt{\lambda_i}x)}{\sinh(\sqrt{\lambda_i}a)} \right. \\ & \left. + 2 \arctan(e^{\sqrt{\lambda_i}a}) - 2 \arctan(e^{\sqrt{\lambda_i}x}) \right) \\ & + u_{0i} \left(e^{-\sqrt{\lambda_i}x} - e^{-\sqrt{\lambda_i}a} \frac{\sinh(\sqrt{\lambda_i}x)}{\sinh(\sqrt{\lambda_i}a)} \right) + u_{ai} \frac{\sinh \sqrt{\lambda_i}x}{\sinh \sqrt{\lambda_i}a}. \end{aligned}$$

This is, in fact, the solution of

$$\begin{cases} -\frac{d^2 u_i}{dx^2}(x) + \lambda_i u_i(x) = f_i, \\ u_i(0) = u_{0i}, \quad u_i(a) = u_{ai}. \end{cases}$$

Now, let us suppose that $\Omega = (0, a) \times (0, b)$ (that is, $\mathcal{O} = (0, b)$). Then,

$$\begin{cases} \lambda_n = \left(\frac{n\pi}{b}\right)^2, \quad n \geq 1 \text{ and } n \in \mathbb{N}, \\ w_n(x_2) = \sqrt{\frac{2}{b}} \sin\left(\frac{n\pi}{b}x_2\right). \end{cases}$$

and

$$u(x, x_2) = \sqrt{\frac{2}{b}} \sum_{n=1}^{\infty} u_n \sin\left(\frac{n\pi}{b}x_2\right).$$

which can be approximated by

$$u^m(x, x_2) = \sqrt{\frac{2}{b}} \sum_{n=1}^m u_n \sin\left(\frac{n\pi}{b}x_2\right),$$

for m large enough.

For instance, if $f \equiv 1$, $u_0 \equiv 1$ and $u_a \equiv 1$, then

$$f_n = u_{0n} = u_{an} = \begin{cases} \frac{2\sqrt{2b}}{n\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Figure 2 (respectively Figure 3) shows the linear interpolation of u^{20} (respectively u^{50}), over a grid of 16 elements for the x -coordinate and 26 elements for the x_2 -coordinate, when $a = 3$ and $b = 5$.

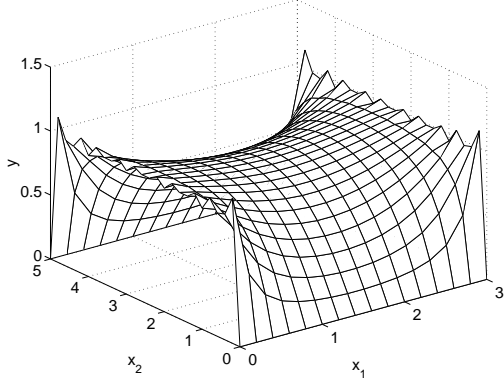


Figure 2: Graph of the function u^{20} of Example 44.

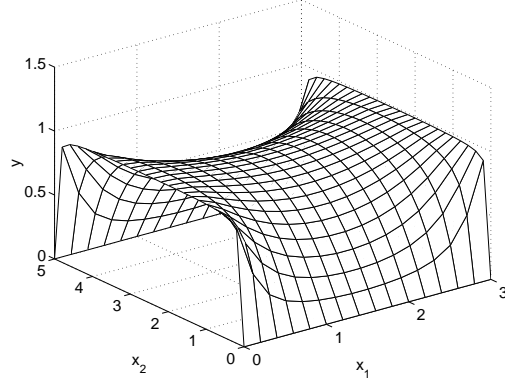


Figure 3: Graph of the function u^{50} of Example 44.

5.3 Neumann Condition on Γ_a

We now change the boundary condition on Γ_a to a Neumann condition. The equation for P and r are unchanged. Only the initial condition for u on Γ_a is changed.

Theorem 45 *Let $j, k, q \in \mathbb{R}$ arbitrary, $u_0 \in V^j$, $f \in L^2(0, a : V^k)$ and $u_a \in V^q$. Then, if $p = \min\{j, k + 3/2, q + 1\}$, there exists a unique solution $u \in \mathcal{C}([0, a] : V^p)$ of the initial value problem*

$$(27) \quad \begin{cases} P\alpha \frac{du}{dx} = u - r, \\ u(a) = P(a)u_a + r(a), \end{cases}$$

where $r \in \mathcal{C}([0, a] : V^{\bar{p}}) \cap L^2(0, a : V^{\bar{p}+1/2}) \cap H^1(0, a : V^{\bar{p}-1/2})$ is the solution of the initial value problem (19) with $\bar{p} = \min\{j, k + 3/2\}$. Further, u is also solution of the problem

$$(28) \quad \begin{cases} -\nabla \cdot (\alpha(x)\nabla u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Sigma, \\ u = u_0 & \text{on } \Gamma_0, \\ \alpha \frac{\partial u}{\partial x} = u_a & \text{on } \Gamma_a \end{cases}$$

and each Fourier coefficient u_i , $i = 1, \dots, \infty$, is given by the formula of Theorem 43, changing u_a by $P(a)u_a + r(a)$.

5.4 Transparent Boundary Conditions

Now we show that the factorization process (2), (3) can be viewed as a Gaussian elimination process. Let us consider the subdomain $\Omega_t^* =]t, a[\times \mathcal{O}$. We can eliminate the unknown function u on the complementary subdomain $\Omega_t =]0, t[\times \mathcal{O}$ by writing a boundary

value problem on Ω_t^* with a boundary condition on Γ_t thanks to the knowledge of $P(t)$ and $r(t)$. In other words this boundary condition which summarizes the behaviour of u on Ω_t is a transparent boundary condition.

Theorem 46 *Let $j, k, q \in \mathbb{R}$ arbitrary, $u_0 \in V^j$, $f \in L^2(0, a : V^k)$ and $u_a \in V^q$. Then, if $p = \min\{j, k + 3/2, q\}$, the unique solution $u \in \mathcal{C}([t, a] : V^p)$ of the initial value problem (24) (or, equivalently, the restriction of the solution of problem (26) to Ω_t^*), where $r \in \mathcal{C}([0, a] : V^{\bar{p}}) \cap L^2(0, a : V^{\bar{p}+1/2}) \cap H^1(0, a : V^{\bar{p}-1/2})$ is the solution of the initial value problem (19) with $\bar{p} = \min\{j, k + 3/2\}$, is also the unique solution of the elliptic boundary value problem*

$$(29) \quad \begin{cases} -\nabla \cdot (\alpha \nabla u) = f & \text{in } \Omega_t^*, \\ u = 0 & \text{on } \Sigma, \\ -P\alpha \frac{\partial u}{\partial x} + u = r & \text{on } \Gamma_t, \\ u = u_a & \text{on } \Gamma_a. \end{cases}$$

Proof. From Corollary (38), the third equation of (29) is satisfied for $x \in [0, a]$ and, therefore, we only need to prove the uniqueness of solution of problem (29). By linearity we suppose that f, r, u_a are equal to zero everywhere and we have to show that the only solution is $u \equiv 0$. Now, the variational formulation is

$$\begin{cases} \text{Find } u \in \mathcal{U}_t = \{\varphi \in H^1(\Omega_t^*) : \varphi|_{\Sigma} = 0, \varphi|_{\Gamma_a} = 0\}, \text{ such that} \\ a(u, \varphi) = 0 \quad \forall \varphi \in \mathcal{U}_t, \end{cases}$$

where

$$a(u, \varphi) = \int_{\Omega_t^*} \alpha \nabla u \nabla \varphi dx dy + \int_{\mathcal{O}} P^{-1}(t) u(t) \varphi(t) dy.$$

Now, since $P(t)$ is a positive definite operator, it is easy to prove, by means of the Poincaré inequality, that $a(\cdot, \cdot)$ is a positive definite bilinear form on \mathcal{U}_t . This concludes the proof by using the Lax-Milgram Theorem (see Section V.3 of [3]). \square

Theorem 47 *Let $j, k, q \in \mathbb{R}$ arbitrary, $u_0 \in V^j$, $f \in L^2(0, a : V^k)$ and $u_a \in V^q$. Then, if $p = \min\{j, k + 3/2, q + 1\}$, the unique solution $u \in \mathcal{C}([0, a] : V^p)$ of the initial value problem (27) (or, equivalently, problem (28)), where r is as in Theorem 46, is also the unique solution of the elliptic problem*

$$\begin{cases} -\nabla \cdot (\alpha \nabla u) = f & \text{in } \Omega_t^*, \\ u = 0 & \text{on } \Sigma, \\ -P\alpha \frac{\partial u}{\partial x} + u = r & \text{on } \Gamma_t, \\ \alpha \frac{\partial u}{\partial x} = u_a & \text{on } \Gamma_a \end{cases}$$

Proof. The proof is analogous to the proof given in Theorem 46. \square

Remark 48 We consider now the problem set on the complementary domain Ω_t .

$$(30) \quad \begin{cases} -\nabla \cdot (\alpha \nabla u) = f & \text{in } \Omega_t, \\ u = 0 & \text{on } \Sigma, \\ u = u_0 & \text{on } \Gamma_0, \\ P\alpha \frac{\partial u}{\partial x} = u - r & \text{on } \Gamma_t. \end{cases}$$

Let us show that this problem does not have a unique solution and therefore it is not well-posed. In fact applying Theorem 45 in Ω_t with an arbitrary Neumann data $h \in V^s$, we have that $P(t)h = P(t)\alpha \frac{\partial u}{\partial x} = u(t) - r(t)$, where $u \in \mathcal{C}([0, t] : V^{s+1})$ is the unique solution of

$$(31) \quad \begin{cases} -\nabla \cdot (\alpha(x) \nabla u) = f & \text{in } \Omega_t, \\ u = 0 & \text{on } \Sigma, \\ u = u_0 & \text{on } \Gamma_0, \\ \alpha \frac{\partial u}{\partial x} = h & \text{on } \Gamma_t. \end{cases}$$

Then for any h , u solution of (31) is solution of (30). By this way we recover the initial definition given in [11], [8] for operator P and function r , before deducing its associated equations.

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